so that

$$f_x = -\frac{3}{2}\nu a U \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin^3\theta \frac{3}{2}$$

and, after evaluating the integrals,

$$f_x = -6\pi\nu a U \tag{20}$$

which is the classical Stokes' law for the sphere.

Coming now to the case of inviscid unsteady flow, i.e., $\nu \to 0$, Eq. (13) provides

$$f_x = \int dV\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}\cdot\nabla\eta_x + \frac{d}{dt}\int_{S_2} dS\boldsymbol{n}\cdot\boldsymbol{b}_2\eta_x \tag{21}$$

For the present case of potential flow, $\boldsymbol{u}=\nabla\phi$, with ϕ solution of the Neumann problem

$$\nabla^2\phi = 0 \tag{22a}$$

$$\boldsymbol{n}\cdot\nabla\phi|_S = \boldsymbol{n}\cdot\boldsymbol{b} \tag{22b}$$

It is then possible to recast Eq. (21) in the form

$$f_x = -\hat{x}\cdot\left(\frac{1}{2}\int_{S_1} dS\boldsymbol{n}|\nabla\phi|^2 + \int_{S_1} dS\boldsymbol{n}\frac{\partial\phi}{\partial t}\right) \tag{23}$$

where the first term is obtained by writing the convection term in Lamb's form and the second term is derived by Green's theorem, taking full advantage of the harmonic character of η_x and ϕ and using the respective boundary conditions Eqs. (11) and (22). Equation (23) coincides with the classical formula for irrotational unsteady flow (see, e.g., Ref. 1, p. 404). If we wish to obtain the drag on the sphere in this case directly from Eq. (21), a simple integration leads to the following final result [the volume integral is easily seen to vanish due to the forward-backward symmetry of the velocity field defined by problem Eq. (22)]

$$f_x = -\frac{2}{3}\pi a^3 \frac{dU}{dt} \tag{24}$$

where the factor multiplying dU/dt is the well-known apparent mass (per unit density) of the sphere (see, e.g., Ref. 2, p. 291).

Conclusion

General formulas for the force and moment acting on a rigid body, immersed in an incompressible flow, have been obtained. Though somewhat difficult to use, insofar as they require the knowledge of the entire solenoidal velocity field, these expressions have a general theoretical interest per se and might be of practical use in applications. For most numerical solutions of flows at intermediate to high Reynolds numbers, the evaluation of the pressure at the body surface gives rise to highly inaccurate results. A more accurate evaluation of the force and moment on the body is possible, therefore, by means of the present formulas, which do not involve the pressure at the body surface, but the velocity field in its entirety.

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Optimum Sensitivity Derivatives of Objective Functions in Nonlinear Programming

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Introduction

OPTIMUM sensitivity analysis is a technique which permits investigation of the sensitivity of an optimization problem's solution to variations of the problem's parameters. It yields derivatives of the optimum values of the design variables and objective function with respect to the parameters. These derivatives may then be used to perform trade-off analyses. Henceforth, these derivatives will be called sensitivity derivatives to distinguish them from the derivatives of objective function and constraints with respect to the design variables that are termed behavior derivatives.

This technique was introduced recently as a tool for structural synthesis.[1] It has been specialized to a formulation of the problem of structural sizing for minimum weight based on approximation concepts and dual methods,[2] and has been used in the optimization of damage tolerant structures.[3] Finally, it is one of the building blocks for a proposed multilevel approach to the design of large engineering systems.[4]

As shown in Ref. 1, a complete sensitivity analysis requires the following quantities as input: 1) the first- and second-behavior derivatives, 2) the first derivatives of behavior and behavior derivatives with respect to the parameters, and 3) the Lagrange multipliers. These need to be evaluated at the optimum point. Among these quantities, the first-behavior derivatives and the Lagrange multipliers are either byproducts of the optimization itself or may be evaluated relatively inexpensively; but, a relatively significant computational cost has to be incurred to calculate the second derivatives. With this as motivation, the present Note shows that the second derivatives may be eliminated completely from the input of the optimum sensitivity analysis, provided that one accepts its curtailment to the first-order sensitivity derivatives of the objective function. Also, it is shown that when a complete first-order sensitivity analysis is performed, second-order sensitivity derivatives of the objective function are available at little additional cost.

First-Order Sensitivity Derivative of the Objective Function

Assume that we start from an optimization problem defined by

$$\min_X F(X,P) \tag{1a}$$

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subject to

$$g(X,P) \leq 0 \tag{1b}$$

where the objective function F and the vector of constraints g are functions of the vectors X containing the n design variables and P containing the k parameters held fixed during the optimization. Using overbars to denote quantities at optimum, we have

$$\bar{F} = \bar{F}(\bar{X}(P),P) \tag{2a}$$

$$\bar{g}_a = \bar{g}_a(\bar{X}(P),P) = 0 \tag{2b}$$

with vector g_a containing only the m constraints that are active at the constrained minimum. The total optimum sensitivity derivative of the objective function with respect to any parameter p is obtained using the chain rule of differentiation.

$$\frac{\mathrm{d}\bar{F}}{\mathrm{d}p} = \bar{F}' + \dot{\bar{F}}^T \bar{X}' \tag{3}$$

where prime symbols indicate partial derivatives with respect to the parameter, while $\dot{F}^T$ is the transposed vector of the derivatives of the objective function F with respect to the design variables (gradient vector). Once the optimum sensitivity derivatives of the variables ($\bar{X}'$) are computed, Eq. (3) gives the optimum sensitivity derivative of the objective function. The following developments describe an approach for obtaining $\mathrm{d}\bar{F}/\mathrm{d}p$ without actually calculating $\bar{X}'$.

The Kuhn-Tucker condition satisfied at a local optimum is

$$\dot{\bar{F}} + \dot{\bar{g}}_a \bar{\lambda} = 0 \tag{4}$$

where the columns of matrix $\dot{g}_a$ are the vectors of derivatives of the active constraints with respect to the design variables (constraints gradient vectors) and vector $\bar{\lambda}$ contains the Lagrange multipliers. Any perturbation of parameter p about its value at the initial optimum must be such that the originally active constraints remain active.

$$\frac{\mathrm{d}}{\mathrm{d}p}\bar{g}_a = \bar{g}_a' + \dot{\bar{g}}_a^T \bar{X}' = 0 \tag{5}$$

Combining Eqs. (4) and (5), we see that

$$\dot{\bar{F}}^T \bar{X}' = \bar{\lambda}^T \bar{g}_a' \tag{6}$$

and

$$\frac{\mathrm{d}\bar{F}}{\mathrm{d}p} = \bar{F}' + \bar{\lambda}^T \bar{g}_a' \tag{7}$$

The Lagrange multipliers may be available as by-products of the optimization scheme, or they may be obtained from

$$\bar{\lambda} = -[\dot{\bar{g}}_a^T \dot{\bar{g}}_a]^{-1} \dot{\bar{g}}_a^T \dot{\bar{F}} \tag{8}$$

then Eq. (7) yields $\mathrm{d}\bar{F}/\mathrm{d}p$ directly. An alternate solution exists because Eq. (5) is the only condition that $\bar{X}'$ must meet in order for Eq. (6) to be satisfied. Therefore, if, for some vector $\bar{Y}'$,

$$\bar{g}_a' + \dot{\bar{g}}_a^T \bar{Y}' = 0 \tag{9}$$

then,

$$\dot{\bar{F}}^T \bar{X}' = \bar{\lambda}^T \bar{g}_a' = \dot{\bar{F}}^T \bar{Y}' \tag{10}$$

Condition (9) is a set of m linear equations in n unknowns and $n \geq m$. We, therefore, may give arbitrary values to $n-m$ components of $\bar{Y}'$ and deduce the remaining m components from Eq. (9). Then,

$$\frac{\mathrm{d}\bar{F}}{\mathrm{d}p} = \bar{F}' + \dot{\bar{F}}^T \bar{Y}' \tag{11}$$

In general, vector $\bar{Y}'$ differs from the true sensitivity derivatives of the variables $\bar{X}'$, unless $n = m$. Assume that, in the space of the design variables X, an infinitesimal perturbation of parameter p results in a move from the initial optimum point in direction $\bar{Y}'$. Then, the resulting design is still at the intersection of the originally active constraints since Eq. (9) is enforced; however, it may not be optimum because Eq. (4) may not be satisfied any more.

The curtailed optimum sensitivity analysis [Eq. (7) or (11)] yields the exact value of $\mathrm{d}\bar{F}/\mathrm{d}p$ at considerably lower cost than the complete analysis described in Ref. 1 since it only involves first-order partial derivatives of F and g_a with respect to the design variables and parameters of the problem. However, it usually does not permit one to obtain $\bar{X}'$. The choice between the two types of analysis is strongly dependent on the application considered and the nature of the optimization problem at hand. It must be guided by a comparison between the cost of calculating second-order partial derivatives of F and g_a, on the one hand, and the benefit of knowing the sensitivity derivatives of the design variables, on the other hand.

Finally, Eq. (7) may be adapted to linear programming problems. Assuming that we have solved

$$\min_X F(X,P) = C(P)^T X \tag{12a}$$

subject to

$$g(X,P) = A(P)X - B(P) \leq 0 \tag{12b}$$

then, from Eq. (7)

$$\frac{\mathrm{d}\bar{F}}{\mathrm{d}p} = \left(\frac{\mathrm{d}}{\mathrm{d}p}C^T\right)\bar{X} + \bar{\lambda}^T\left(\frac{\mathrm{d}}{\mathrm{d}p}A\right)\bar{X} - \bar{\lambda}^T\left(\frac{\mathrm{d}}{\mathrm{d}p}B\right) \tag{13}$$

a result similar to one given in Ref. 5 but obtained here in a significantly more direct way.

Second-Order Sensitivity Derivatives of the Objective Function

Using once again the chain rule of differentiation, we may obtain from Eq. (7) a second-order sensitivity derivative of the objective function.

$$\frac{\mathrm{d}^2\bar{F}}{\mathrm{d}p^2} = \bar{F}'' + \dot{\bar{F}}'^T \bar{X}' + \bar{\lambda}'^T \bar{g}_a' + \bar{\lambda}^T(\bar{g}_a'' + \dot{\bar{g}}_a' \bar{X}') \tag{14}$$

where F'' and g_a'' are second partial derivatives of F and vector g_a with respect to parameter p, $\dot{F}'$ is the first partial derivative of the gradient vector of F, $\dot{g}_a'$ is the matrix of the first partial derivatives of the gradients of the active constraints, and λ' is the vector of the sensitivity derivatives of the Lagrange multipliers. This result was first presented in Ref. 6. It is quite interesting since it shows that when a complete first-order sensitivity analysis is performed, little additional calculation is needed to obtain the second sensitivity derivative of F. A first-order sensitivity analysis requires the calculation of $\dot{\bar{F}}'$, $\bar{g}_a'$, $\dot{\bar{g}}_a'$, and, of course, yields $\bar{X}'$ and $\bar{\lambda}'$. Then, only $\bar{F}''$ and $\bar{g}_a''$ must be found to calculate $\mathrm{d}^2\bar{F}/\mathrm{d}p^2$. Similar developments may be made to find cross derivatives of the type $\mathrm{d}^2\bar{F}/\mathrm{d}p_i\mathrm{d}p_j$ where p_i and p_j are two different parameters of the problem.

This information may be used to obtain a quadratic extrapolation formula relating the optimum objective function of problem (1) to parameter p (or to parameters p_i and p_j). In principle, such a quadratic extrapolation may be more accurate than the linear approximation proposed in Ref. 1. However, a word of caution is necessary. It was shown in Ref. 7 (and also in Ref. 2) that the quality of extrapolations based on first-order sensitivity derivatives may be reduced greatly when parameter variations cause changes in the membership of the set of the active constraints. If such changes occur, the range of accurate extrapolation may not be extended by use of quadratic extrapolation based on $d^2\bar{F}/dp^2$.

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Vorticity at the Shock Foot in Inviscid Flow

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Introduction

IT is characteristic of transonic flows to have a shock or shocks embedded in the flowfield. The flow immediately behind the shock is related to the flow ahead of it by the Rankine-Hugoniot conditions and is a function of the shock shape. If the shock is normal to the body surface, the flow behind the shock will be subsonic and its shape will, in general, not be known a priori. The shock shape must be determined in conjunction with the local flowfield.

Although inviscid transonic flow past a body can be computed routinely using numerical methods developed in the last decade, the shock region has always been the most erroneous part of the solution. Often the potential approximation is made, and this is inconsistent with the conservation of normal momentum across the shock. Also, only the weak form of the governing equation is satisfied to an order normally so restricted by numerical stability that the actual shock is smeared over several grid points. As the shock gets stronger, entropy and vorticity production behind the shock can no longer be ignored and potential theory fails. These effects, added to the already complex flowfield, makes the construction of a proper numerical scheme a difficult task.

In this Note we study the flowfield immediately downstream of a shock at its root where the shock meets a smooth convex surface. Lin and Rubinov[1] first noted that a singularity occurs at the shock root. They also argued that the shock shape at the root must be of the form

$$\xi = k\eta^{3/2}$$

where ξ, η are coordinates of the shock measured along and normal to the body, respectively. Zierep[2] also found the same shock shape but was unable to determine the constant k for a convex body. Later Gadd[3] pointed out that the flow behind and at the shock root, determined by the Rankine-Hugoniot conditions, experiences a discontinuity in curvature in order to conform to the body. Such a flow is known to have a multivalued normal pressure gradient and a streamwise pressure gradient that is logarithmically singular.

We shall determine the vorticity behind the curved shock at this singular point and discuss to what extent the flowfield is affected by this vorticity.

Shock-Induced Vorticity

It is well known that the vorticity behind a shock can be computed by applying Crocco's theorem, i.e.,

$$\zeta_2 = -\frac{1}{q_2 \sin(\sigma-\alpha)}\left[\frac{1}{2}\frac{dq_2^2}{d\ell} + \frac{1}{\rho_2}\frac{dp_2}{d\ell}\right] \tag{1}$$

where subscript 2 denotes quantities evaluated after the shock, ζ is the vorticity induced by the shock, q the flow speed, ρ the density, p the pressure, ℓ the distance along the shock, σ the shock angle measured relative to the upstream flow, and α the flow deflection angle after the shock (Fig. 1).

The Rankine-Hugoniot conditions require that the post-shock quantities be related to preshock quantities as follows:

$$q_2^2 = q_1^2[1-(1-\epsilon^2)\sin^2\sigma]$$

$$p_2 = p_1 + \rho_1 q_1^2(1-\epsilon)\sin^2\sigma$$

$$\epsilon = \frac{\rho_1}{\rho_2} = \frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M_1^2\sin^2\sigma} \tag{2}$$

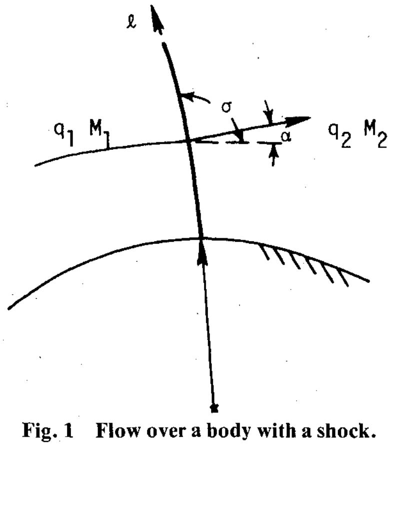

Fig. 1 Flow over a body with a shock.

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Dedicated to Professor William R. Sears in celebration of his 70th birthday, with the author's admiration.